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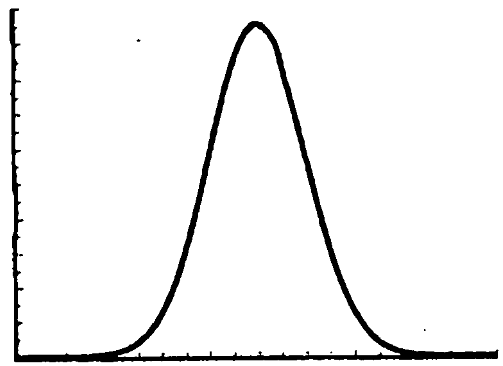
Figure 4.6
The Normal Distribution



values of R, U, and D adjust automatically as we consider more and more periods during the fixed calendar interval t. The absolute value of each becomes smaller, exactly as we would expect for a shorter time period. The values for U and D depend on the riskiness of the stock returns. The probability of a stock price increase depends upon the mean return on the stock. Thus, if we can estimate the standard deviation of stock returns, we have reasonable inputs to the binomial model. These can replace the arbitrary example values that we have been using.[4]

The Black-Scholes Option Pricing Model

To this point, we have developed the binomial option pricing model. We have discussed the log-normal distribution of stock returns and have presented up and down factors for the binomial model that are consistent with the log-normal distribution of stock returns. Also, we have seen how to adjust the precision of the binomial model by dividing a given unit of calendar time into more and more periods. As we deal with more periods, however, the calculations in the binomial

model become cumbersome. As the number of periods in the binomial model becomes very large, the binomial model converges to the famous Black-Scholes Option Pricing Model.

Fischer Black and Myron Scholes developed their option pricing model under the assumptions that asset prices adjust to prevent arbitrage, that stock prices change continuously, and that stock returns follow a log-normal distribution.[5] Also, their model holds for European call options on stocks with no dividends. Further, they assume that the interest rate and the volatility of the stock remain constant over the life of the option. The mathematics they used to derive their result include stochastic calculus, which is beyond the scope of this text. In this section, we present their model and illustrate the basic intuition that underlies it. We show that the form of the Black-Scholes Model parallels the bounds on option pricing that we have already observed. In fact, the form of the Black-Scholes model is very close to the binomial model we have just been considering.

The Black-Scholes Call Option Pricing Model

The following expression gives the Black-Scholes Option Pricing Model for a call option:

$$C = SN(d_1) - Ee^{-rt}N(d_2) \qquad 4.7$$

where:

$N() =$ *cumulative normal distribution function*

$$d_1 = \frac{\ln(\frac{S}{E}) + (r + .5\sigma^2)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

This model has the general form we have long considered—the value of a call must equal or exceed the stock price minus the present value of the exercise price:

$$C \geq S - Ee^{-rt}$$

To adapt this formula to account for risk, as in the Black-Scholes Model, we multiply the stock price and the exercise price by some factors to account for risk, giving the general form:

$$C = S \times \textit{Risk Factor \#1} - Ee^{-rt} \times \textit{Risk Factor \#2}$$

The binomial model shares this general form with the Black-Scholes model. With the binomial model, the risk adjustment factors were the large bracketed expressions of equation 4.2. With the Black-Scholes model, the risk factors are $N(d_1)$ and $N(d_2)$. In the Black-Scholes model, these risk adjustment factors are the continuous time equivalent of the bracketed expressions in the Binomial model.

The Black-Scholes Call Option Pricing Model

$$C = SN(d_1) - Ee^{-rt}N(d_2)$$

Computing Black-Scholes Option Prices

In this section, we show how to compute Black-Scholes option prices. Assume that a stock trades at \$100 and the risk-free interest rate is 6 percent. A call option on the stock has an exercise price of \$100 and expires in one year. The standard deviation of the stock's returns is .10 per year. We compute the values of d_1 and d_2 as follows:

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + (r + .5\sigma^2)t}{\sigma\sqrt{t}}$$

$$= \frac{\ln\left(\frac{100}{100}\right) + [.06 + .5\ (.01)\ ]1}{.1\sqrt{1}} = .65$$

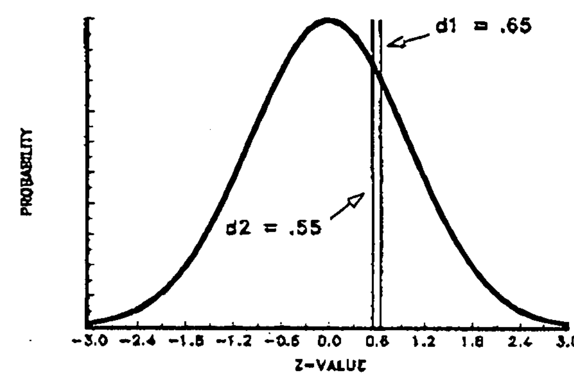
$$d_2 = d_1 - \sigma\sqrt{t}$$
$$= .65 - .1 \times 1 = .55$$

Next, we find the cumulative normal values associated with d_1 and d_2. These values are the probability that a normally distributed variable with a zero mean and a standard deviation of 1.0 will have a value equal to or less than the d_1 or d_2 term we are considering. Figure 4.7 shows a graph of a normally distributed variable with a zero mean and a standard deviation of 1.0. It shows the values of d_1 and d_2 for our example. For illustration, we focus on d_1, which equals .65. In finding $N(d_1)$, we want to know which portion of the area under the curve lies to the left of .65. This is the value of $N(d_1)$. Clearly, the value we seek is larger than .5, because d_1 is above the mean of zero. We can find the exact value by consulting a table of the cumulative normal distribution for this variable. We present this table as Appendix A. Also, we can use **OPTION!** to find these values. For a value of .65 drawn from the border of the table, we find our probability in the interior: $N(.65) = .7422$. Similarly, $N(d_2) = N(.55) = .7088$. We now have:

$$C = \$100 \times .7422 - \$100 \times .9418 \times .7088 = \$7.46$$

We chose these values for our example because they parallel the values from our original binomial example. There we also assumed that the stock traded for \$100 and that the risk-free rate was 6 percent. We assumed an up factor of 10 percent and a down factor of -10 percent. With a single-period binomial model and these values, we found that a call must be priced at \$7.55. The two results are close. However, if we use more periods in the binomial model and

Figure 4.7
The Standardized Normal Distribution



use up and down factors that are consistent with a log-normal distribution of stock returns, the binomial model will converge to the Black-Scholes model price. The reader can explore this possibility by using OPTION!.

The Hedge Ratio

In our arbitrage discussions, we found that we could form a portfolio one period before expiration that would mimic a call option. With just one period until expiration, we saw that:

$$C = NS - Ee^{-rt}$$

In this equation, we mimic the call by borrowing the present value of the exercise price and by holding N shares. In this context, N is a hedge ratio, the number of shares to hold to form the mimicking portfolio. By holding N shares along with our borrowing, we create a portfolio that has the same payoffs as the call. If we hold the call and sell the mimicking portfolio, the entire set of investments will be worth zero at expiration. We will be perfectly hedged against any profits or losses.

A similar analysis applies to the Black-Scholes Model. For the Black-Scholes Model, $N(d_1)$ is the hedge ratio. However, the Black-Scholes Model operates in continuous time. Thus, we treat the hedge as holding for the next instant or for the next infinitesimal stock price change. If the stock price changes in the next instant, we will be hedged. However, a changing stock price will generally change the hedge ratio. Thus, $N(d_1)$ is an instantaneous hedge ratio. It holds for the next instant, after which we may need to slightly rebalance our mimicking portfolio to be fully hedged.

The hedge ratio also has considerable practical importance because it shows how the call price will change for a one unit change in the stock price. Thus, if we hold a call option on one share and sell $N(d_1)$ shares of stock, the resulting option/stock portfolio will not change in value when the stock price changes. As we will see, this principle holds for the next instant and for infinitesimal changes in stock prices. The resulting portfolio is risk-free, because its price does not change as the stock price changes. Therefore, the portfolio must earn the risk-free rate on the invested funds, which would equal $C - N(d_1) \times S$.

Because the hedge ratio relates the change in the price of the call to a change in the price of the stock, $N(d_1)$ is also known as the delta for the call. Earlier we computed a Black-Scholes price of $7.46 for our example data. There we found that $N(d_1) = .7422$. Based on this value, we expect an increase in the price of the call if the stock price rises by $1. If the stock price rose to $101, the Black-Scholes call price would be $8.22, for a $.76 difference. This is almost, but not quite, equal to the $.74 we expect. There is a reason for this difference.

The Black-Scholes Model operates on the assumption that stock prices can change at every instant. However, they can change by only a very small amount. We assumed an instantaneous one percent change. With the Black-Scholes Model, a one percent change would be accomplished by numerous smaller changes. Each of these smaller changes would require a re-balancing of the mimicking portfolio. For this reason, the hedge ratio gives only the approximate change in the price of the call if we assume a large change in the price of the stock. Nonetheless, it offers a very good approximation.

The Black-Scholes Put Option Pricing Model

Black and Scholes developed their option pricing model for calls only. However, we can find the Black-Scholes Model for European puts by applying put-call parity:

$$P = C - S + Ee^{-rt}$$

Substituting the Black-Scholes call formula in the put-call parity equation gives:

$$P = SN(d_1) - Ee^{-rt}N(d_2) - S + Ee^{-rt}$$

Collecting like terms simplifies the equation to:

$$P = S\left[N(d_1) - 1\right] + Ee^{-rt}\left[1 - N(d_2)\right]$$

If we consider the cumulative distribution of all values from $-\infty$ to $+\infty$, the maximum value is 1.0. For any value of d_1 we consider, part of the whole must lie at or below the value and the remainder must lie above it. For example, if $N(d_1)$ is .7422, for $d_1 = .65$, then .2578 of the total area under the curve must lie at values greater than .65. Now we apply a principle of normal distributions. The normal distribution is symmetrical, so the same percentage of the area under the curve that lies above d_1 must lie below $-d_1$. Therefore, for any symmetrical distribution and any arbitrary value, w:

$$N(w) + N(-w) = 1$$

Following this pattern and substituting for $N(d_1)$ and $N(d_2)$ gives the equivalent Black-Scholes value for a put option.

$$P = Ee^{-rt}N(-d_2) - SN(-d_1) \qquad 4.8$$

This equation has the familiar form that we have been exploring since Chapter 2. We emphasize that the Black-Scholes Model for puts holds only for European puts.

The Black-Scholes Put Option Pricing Model [illegible]

Inputs for the Black-Scholes Model

We have seen that the Black-Scholes Model for the price of an option depends on five variables: the stock price, the exercise price, the time until expiration, the risk-free rate, and the standard deviation of the stock. Of these, the stock price is observable in the financial press or on a trading terminal. The exercise price and the time until expiration can be known with certainty. We want to consider how to obtain estimates of the other two parameters; the risk-free interest rate and the standard deviation of the stock.

Estimating the Risk-Free Rate of Interest

Estimates of the risk-free interest rate are widely available and are usually quite reliable.[6] There are still a few points to consider, however. First, we need to select the correct rate. Because the Black-Scholes Model uses a risk-free rate, we can use the Treasury-bill rate as a good estimate. Quoted interest rates for T-bills are expressed as discount rates. We need to convert these to regular interest rates and express them as continuously compounded rates. As a second consideration, we should select the maturity of the T-bill carefully. If the yield curve has a steep slope, yields for different maturities can differ significantly. With T-bills maturing each week, we choose the bill that matures closest to the option expiration.

We illustrate the computation with the following example. Consider a T-bill with 84 days until maturity. Its bid yield is 8.83 and its asking yield is 8.77. Letting B and A be the bid and asked yields, the following formula gives the price of a T-bill as a percentage of its face value:

$$\begin{aligned} P_{TB} &= 1 - .01\left(\frac{B+A}{2}\right)\left(\frac{Days\ Until\ Maturity}{360}\right) \\ &= 1 - .01\left(\frac{8.83+8.77}{2}\right)\left(\frac{84}{360}\right) \\ &= .97947 \end{aligned}$$

In this formula, we average the bid and asked yields to estimate the unobservable true yield which lies between the observable bid and asked yields. For our example, the price of the T-bill is 97.947 percent of its face value. To find the corresponding continuously compounded rate, we solve the following equation for r:

$$\begin{aligned} e^{rt} &= \frac{1}{P_{TB}} \\ e^{r(.23)} &= 1/.97947 \\ .23r &= \ln(1.02096) = .0207 \\ r &= .0902 \end{aligned}$$

In the equation, $t = .23$ because 84 days is 23 percent of a year. Thus, the appropriate interest rate in this example is 9.02 percent.

Securing good estimates of the risk-free interest rate is fairly easy. However, having an exact estimate is not critical. As we show later in this chapter, the option price is not very sensitive to the interest rate.

Estimating the Stock's Standard Deviation

Estimating the standard deviation of the stock's returns is more difficult and more important than estimating the risk-free rate. The Black-Scholes Model takes as its input the current, instantaneous standard deviation of the stock. In other words, the immediate volatility of the stock is the riskiness of the stock that affects the option price. The Black-Scholes Model also assumes that the volatility is constant over the life of the option.[7] There are two basic ways to estimate the volatility. The first method uses historical data, while the second technique employs fresh data from the options market itself. This second method uses option prices to find the option market's estimate of the stock's standard deviation. An estimate of the stock's standard deviation that is drawn from the options market is called an implied volatility. We consider each method in turn.[8]

Historical Data—To estimate volatility using historical data, we compute the price relatives, logarithmic price relatives, and the mean and standard deviation of the logarithmic price relatives. Letting PR_t indicate the price relative for day t so that $PR_t = P_t/P_{t-1}$, we give the formulas for the mean and variance of the logarithmic price relatives as follows:

$$\overline{PR} = \frac{1}{T}\sum_{t=1}^{T} \ln PR_t$$

$$VAR\ (PR) = \frac{1}{T-1}\sum_{t=1}^{T} \left(\ln PR_t - \overline{PR}\right)^2$$

As an example, we apply these formulas to data in the following table, which gives eleven days of price information for a stock. With eleven price observations, we compute ten daily returns. The first column tracks the day, while the second column records the stock's closing price for the day. The third column computes the price relative from the prices in column 2. The fourth column gives the log of the price relative in column 3. The last column contains the result of subtracting the mean of the logarithmic price relatives from each observation and squaring the result.

Day	P_t	PR_t	$\ln(PR_t)$	$[\ln PR_t - \overline{PR}_t]^2$
0	100.00			
1	101.50	1.0150	0.0149	0.000154
2	98.00	0.9655	-0.0351	0.001410
3	96.75	0.9872	-0.0128	0.000234
4	100.50	1.0388	0.0380	0.001264
5	101.00	1.0050	0.0050	0.000006
6	103.25	1.0223	0.0220	0.000382
7	105.00	1.0169	0.0168	0.000205
8	102.75	0.9786	-0.0217	0.000582
9	103.00	1.0024	0.0024	0.000000
10	102.50	0.9951	-0.0049	0.000053
		Sums	.0247	0.004294

Sample μ = .0247/10 = .00247
Sample σ^2 = .004294/9 = .000477
Sample σ = .021843

The mean, variance, and standard deviation that we have calculated are all based on our sample of daily data. We use the sample standard deviation as an input to the Black-Scholes Model.

Three inputs to the Black-Scholes Model depend on the unit of time. These inputs are the interest rate, the time until expiration, and the standard deviation. We can use any single measure we wish, but we need to express all three variables in the same time units. For example, we can use days as our time unit and express the time until expiration as the number of days remaining. Then we must also use a daily estimation of the standard deviation and the interest rate for a single day. Generally, one year is the most convenient common unit of time. Therefore, we need to convert our daily standard deviation into a comparable yearly estimate. We have estimated our daily standard deviation of ten days. However, these are ten trading days, not calendar days. Accordingly, we recognize that we are working in trading time, not calendar time. Deleting weekend days and holidays, each year has about 250-252 trading days. We use 250 trading days per year.

We have already seen that stock prices are distributed with a standard deviation that increases as the square root of time. Accordingly, we can adjust the time dimension of our volatility estimate by multiplying it by the square root of time. For example, we convert from our daily standard deviation estimate to an equivalent yearly value by multiplying the daily estimate times the square root of 250.[9]

$$\textit{Annualized } \sigma = \textit{Daily } \sigma \times \sqrt{250}$$

For our daily estimate of .021843, the estimated standard deviation in annual terms is .3454.

In our example, we have used ten days of data. In actual practice, we face a trade-off between using the most recent possible data and using more data. In statistics, we almost always get more reliable estimates by using more data. However, the Black-Scholes Model takes the instantaneous standard deviation as an input. This gives great importance to using current data. If we use the last year of historical data, then we have a rich data set for estimating the old volatility. Using just ten days, as we did in our example, emphasizes current data, but it is really not very much data for getting a reliable estimate.

To emphasize the importance of using current data, consider the Crash of 1987. On Bloody Monday, October 19, 1987, the market lost about 22 percent of its value. If we used a full year of daily data to estimate a stock's historical volatility the next day, our estimate would be too low. In the light of the Crash, the instantaneous volatility had surely increased.

Implied Volatility—To overcome the limitations inherent in using historical data to estimate standard deviations, some scholars have turned to techniques of implied volatility. In this section, we show how to use market data and the Black-Scholes Model to estimate a stock's volatility. There are five inputs to the Black-Scholes Model, which the model relates to a sixth variable, the call price. With a total of six variables, any five imply a unique value for the sixth. The

technique of implied volatility uses known values of five variables to estimate the standard deviation. The estimated standard deviation is an implied volatility because it is the value implied by the other five variables in the model.[10]

To find implied volatilities, we begin with established values for the stock price, exercise price, interest rate, time until expiration, and the call price. We use these to find the implied standard deviation. However, the standard deviation enters the Black-Scholes Model through the values for d_1 and d_2, which are used to determine the values of the cumulative normal distribution. As a result, we cannot solve for the standard deviation directly. Instead, we must search for the volatility that makes the Black-Scholes equation hold. To do this, we need a computer. Otherwise, we would have to try an estimate of the standard deviation, make all of the Black-Scholes computations by hand, and adjust the standard deviation for the next try. This would be cumbersome and time consuming. Therefore, implied volatilities are almost always found using a computer. **OPTION!** has a module for finding implied volatilities.

For most stocks with options, several options with different expirations trade at once. Some researchers have argued that all of these options should be used to find the volatility implied by each. The resulting estimates are then given weights and averaged to find a single volatility estimate. The single estimate is known as a weighted implied standard deviation. In principle, this is a good idea because it uses more information. Other things being equal, estimates based on more information should dominate estimates based on less information. However, some options trade infrequently, which makes their prices less reliable for computing implied volatilities. In addition, options way out-of-the-money give somewhat spurious volatility estimates. Virtually all weighting schemes give the highest weight to options closest to-the-money. At-the-money options tend to give the least biased volatility estimates, and many option traders derive implied volatilities by focusing on at-the-money options.[11]

Consider the following example of an implied standard deviation based on a call option. We assume that $E = \$100$ and the option is at-the-money, so $S = \$100$. We also assume that the option has 90 days remaining until expiration and that the risk-free interest rate is 10 percent, so we have $t = 90$ and $R_f = .10$. The call price is \$5.00. To find the implied standard deviation, we need to find the standard deviation that is consistent with these other values. To do this, we can compute the Black-Scholes Model price for alternative standard deviations. We adjust the standard deviation to make the option price converge to its actual price of \$5.00. The sequence of standard deviations and call prices below shows this relationship. In our example, we first try $\sigma = .1$, which gives a call price of \$3.41. This price is too low. Thus, we know the correct standard deviation must be larger, because the call price varies directly with the standard deviation. Next, $\sigma = .5$ results in a call price of \$11.03, which is too high. Now we know that the standard deviation must be greater than .1, but less than .5. The task is to find the standard deviation that gives a call value equal to the specified \$5.00. This

happens with $\sigma = .187$. Using the implied volatility module of **OPTION!**, we find that the exact standard deviation is .186800.[12]

Standard Deviation	Corresponding Call Price	
.1	\$ 3.41	too low
.5	11.03	too high
.3	7.16	too high
.2	5.24	too high
.15	4.31	too low
.175	4.78	too low
.18	4.87	too low
.185	4.97	too low
.19	5.06	too high
.188	5.02	too high
.187	5.00	success

Sensitivity of the Black-Scholes Model

We have seen that the Black-Scholes Model expresses the value of a call as a function of five factors: the stock price, the exercise price, the time until expiration, the risk-free rate, and the volatility of the stock. Also, we have just seen that $N(d_1)$ shows how the call price changes for a change in the stock price. In this section, we explore the sensitivity of call prices to the five parameters in the Black-Scholes Model.

Sensitivity of Call Prices to the Stock Price—The Delta

Figure 4.8 graphs the relationship between the call price and the stock price. In this graph, we hold the other factors constant. Specifically, we assume that the exercise price is \$100, the risk-free rate is .06, the option expires in one year, and the standard deviation of the stock returns is .1. The curved line shows the value of the call for various stock prices. When the stock price is \$100, we know that the call price is \$7.46, for example.

The slope of the curved line in Figure 4.8 shows the sensitivity of the call's price to a change in the stock price. In fact, the slope of the line at any point equals the call's delta, $N(d_1)$. The delta is positive through the whole line—an increasing stock price always gives an increasing call price. However, when the stock price is low relative to the exercise price, the curved line is very flat. This shows that the call price is not very responsive to stock price changes if the call